The Square Root Approximation to the Dispersion Relation of the Axially-symmetric Electron Wave on a Cylindrical Plasma

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This paper suggests the use of a simple square root approximation to the dispersion relation of axially-symmetric electron surface waves on cylindrical plasmas. The point is not merely to substitute the exact expression for the dispersion relation which involves a number of Bessel functions with a more tractable analytical approximant, but to cast the dispersion relation in a form useful in the comparison with other waves, such as water surface gravity waves and the associated tide-rip effect. The square root form of the dispersion relation is also of help in the analysis of surfactron plasmas, as it directly predicts a linear roll-off of electron density in the discharge.

Introduction

Electron waves on a plasma guide attracted the attention of investigators for a long time. After the pioneering papers by Trivelpiece and Gould [1] and Akao and Ida [2], this field has seen very important developments [3], boosted by the discovery of the surfatron, a surface-wave discharge. Since then the publication rate has not declined [4]. The phase characteristics of various modes have been analyzed in detail, in particular those of the axially-symmetric waves [5].

In [6] it was shown how to simplify the quasi-static dispersion relation of an axially-symmetric surface wave by the introduction of an effective permittivity of an equivalent dielectric around the plasma, which virtually reduces the finite glass thickness case to the case of a plasma in a homogeneous infinite dielectric. It is this simplified theory which we use later in this paper to prove the square root approximation in the region of small normalized wavenumbers in the form $\omega/\omega_p = \text{const} \cdot \sqrt{\beta a}$, where ω is the wave radian frequency, ω_p the plasma radian frequency, β the wavenumber, and a the tube radius with $\beta a < 1$. The remarkable simplicity of this approximation induced us to investigate its usefulness from various aspects. To the best of our knowledge the relatively simple structure of this expression and its efficiency in applications have never been noticed before. Let us remark immediately that the limitation $\beta a < 1$ does not diminish the applicability of the square root expression, as most relevant experiments are conducted in this range (see [7]), and the model of a homogeneous column [8] is also consistent with it.

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Graphical Test

The dispersion relation of the axially-symmetric electron wave on a cylindrical, homogeneous and collisionless plasma in a glass tube is well known (compare [6]). The quasi-static approximation can be written as

$$1 - \frac{\omega_p^2}{\omega^2} - \varepsilon_g \frac{I_0(x)}{I_1(x)} \cdot \frac{(1-\varepsilon_g) I_1(x) K_0(z) K_1(z) - K_1(x) S(z)}{(1-\varepsilon_g) I_0(x) K_0(z) K_1(z) + K_0(x) S(z)} = 0, \quad (1)$$

where $S(z) = -I_0(z) K_1(z) - \varepsilon_g K_0(z) I_1(z)$, and I and K are modified Bessel functions of the first and second kind of order 0 and 1, as denoted by indices, and of arguments $x = \beta a$ and $z = \beta b$, with $\beta = 2\pi/\lambda$ the phase coefficient (or wavenumber), a the inner and b the outer radius of the glass tube surrounding the plasma, and ε_g the permittivity of the glass.

Figure 1 shows a dispersion relation of the axially-symmetric electron surface wave in logarithmic scale. The computation is for $\varepsilon_g = 4.8$ and $b/a = 1.25$. A best fit straight line of the type $\ln(\omega/\omega_p) = \varrho \ln(x) + q$ is also shown, where $\varrho = 0.5691$ and $q = -0.8634$. In other words, the square-root approximation is of the type $\omega/\omega_p = 0.42\, x^{0.57}$.

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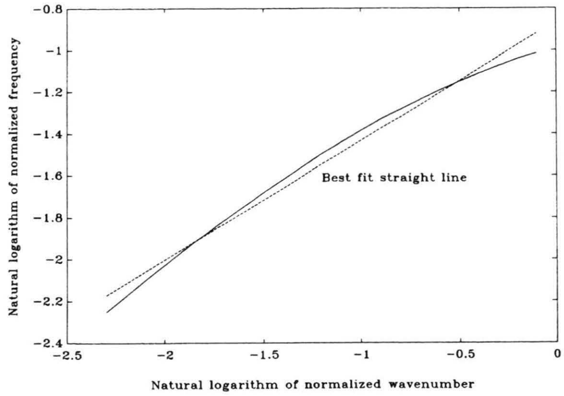

Fig. 1. The natural logarithm of the normalized frequency (the ratio ω/ω_p) vs. the natural logarithm of normalized wavenumber (or phase coefficient) βa. The best fit straight line corresponds to a slope of 0.5387.

We see that the departures from the square-root law are small and could even be smaller if the interval of x is further reduced, which is generally permissible [9].

Analysis

As it is a lengthy procedure to seek an approximate form of (1) for $x < 1$, we depart from the dispersion relation valid for a very thick glass wall

$$\frac{\omega}{\omega_p} = \frac{1}{\sqrt{1 - M(x)\,\varepsilon}}, \tag{2}$$

where $M(x) = -I_0(x)K_1(x)/[I_1(x)K_0(x)]$. According to [6], the following choice of the effective permittivity of the equivalent dielectric is to be made:

$$\varepsilon = 1 + (\varepsilon_g - 1)\tanh\left[\left(\frac{b}{a} - 1\right)x\right]. \tag{3}$$

For small arguments we set $\tanh(x) \simeq x$, as for $x = 1$ we have $(b/a - 1)x \simeq 0.2$. This gives

$$\varepsilon \simeq 1 + x. \tag{4}$$

Further, we have approximately $I_0(x) \simeq 1$ and $I_1(x) \simeq 0.5\,x$ (see [10]). The ratio $K_1/K_0 \equiv f(x)$ is well approximated by $f(x) = 3/(1 + x)$ (see [11]), which then gives $\varepsilon M(x) \simeq 6/x$, and from (2) $\omega/\omega_p \simeq \sqrt{x/(6 + x)}$, and in fact

$$\frac{\omega}{\omega_p} \simeq 0.4\sqrt{x}. \tag{5}$$

We see that the square-root approximation really follows from the dispersion relation by simplification of the analytical expressions.

Comparison with Experiment

Figure 2 shows a comparison of the square-root approximation with experimental data. The results were obtained in a cylindrical gas discharge plasma in mercury vapor. The outer and inner glass radii are $b = 7.5$ mm and $a = 6.0$ mm, which gives $b/a = 1.25$. The permittivity of the glass tube is $\varepsilon = 4.8$. The signal frequencies range from 110 MHz to 250 MHz, and the arc discharge currents is 30 mA. A detailed description of the apparatus and measuring techniques is to be found in [12].

The best fit line in Fig. 2 corresponds to the equation

$$\frac{\omega}{\omega_p} \simeq 0.39\sqrt{x}, \tag{6}$$

in good agreement with the result (5).

Applications and Comparison with other Wave Phenomena

Although the square-root approximation appears in a number of problems, such as the low-frequency surface waves on nonisothermal plasma columns [13],

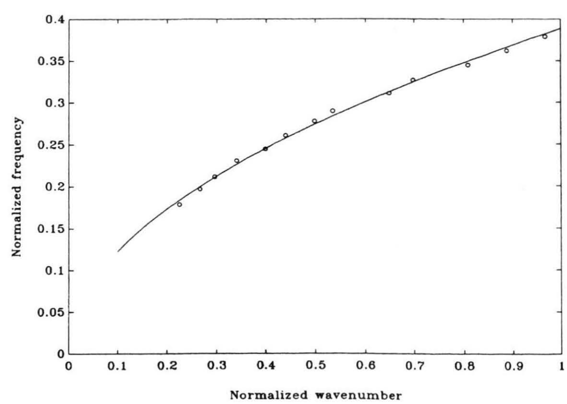

Fig. 2. Experimental points at 30 mA arc current in a band of signal frequencies from 110 MHz to 250 MHz denoted by circles. The solid line is $\omega/\omega_p = 0.39\sqrt{\beta a}$. The normalized frequency is ω/ω_p, the normalized wavenumber (or phase coefficient) is βa.

we restrict ourselves here to two cases where the square-root approximation must be welcome.

Surfatron Plasma

It is known that cylindrical plasma columns can be obtained by an electromagnetic surface wave. The energy of the wave results in ionization of the gas medium, and the wave creates the medium along which it propagates. Plasma sources which function on this principle are known as surfatrons [14]. Due to their stability and reproducibility, surfatron plasmas were both used and studied in great detail in recent years. It is known that the plasma density in the central part of the surfatron discharge exhibits a linear decay with the axial coordinate z (see [15]). According to [16], the plasma density $n(z)$ is obtained by solving the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{\frac{n(z)}{\alpha(z)}\right\} = -2n(z)\,, \tag{7}$$

where $\alpha(z)$ is the local attenuation coefficient of the wave. This coefficient is given by (compare with [17])

$$\alpha(z) = \frac{\nu}{2\,v_g\,\omega_p\,a}\,, \tag{8}$$

where ν is the electron collision frequency for momentum transfer and v_g the normalized wave group velocity. From the square-root approximation (5), we have immediately $\alpha = \text{const} \cdot n^{-1}$, and this yields for the solution of (7) $\mathrm{d}n/\mathrm{d}z = \text{const.}$, in accordance with experiment and all the evidence on surfatron plasmas. This encourages the use of the square-root approximation in more involved expressions for the attenuation coefficient [18].

The Tide-rip Effect

An interesting wave phenomenon, named the tide-rip effect in the English language, occurs in straits. A popular description is to be found in [19]. Although important progress has been made [20], the details of the tide-rip effect are not wholly understood. The key idea seems to be the transformation of deep water waves in the presence of a horizontally inhomogeneous flow [21]. It is well known that the dispersion relation of gravity surface waves on deep water is given by $\omega = \sqrt{g\,\beta}$, where g is the acceleration due to gravity [22]. As this relation is also of the square-root type akin to (5), we were led to believe that some analogy with plasma waves could help in the tide-rip problem. This is supported by previous experiments [23], and also computations [24]. This work is in progress.

Other Approximative Forms of the Dispersion Relation

The reader was warned in advance that the main aim of this paper is not to produce an accurate approximation to the dispersion relation of the $n = 0$ electron wave on a plasma column. There are certainly very many other analytical expressions which are numerically better then the simple square-root expression, and also others which would serve better in other problems. For example, the theoretical dispersion relation in Fig. 1 is fitted with a second degree polynomial in the least squares sense, and the resulting curve falls within the thickness of the line of the graph. Very many other simple expressions are eligible, such as power laws (with a power closer to 2/3 than 1/2), hyperbolic tangents and the like. Of these we would like to mention one published by K. E. Lonngren et al. [25], of the type

$$D(\omega, \beta) = 1 - \omega^2/\omega_0^2 + k^2/k_0^2 = 0\,,$$

applicable to very many waves, including the axially-symmetric plasma surface wave. In particular, a simplification of the above expression is

$$\omega(\beta) = C_1\,\beta + C_3\,\beta^3$$

and was found useful in the study of the propagation of pulses of both ion sound and electron waves, where it leads to expressions including Airy functions.

Conclusions

In the study of axially-symmetric electron waves on plasma columns the homogeneous plasma model works well when the normalized phase coefficient is less than unity. It is in this region that most experiments are carried out, as the attenuation under this condition is not excessive. Therefore, the dispersion relation of the axially-symmetric wave $\omega/\omega_p = f(\beta a)$ is of importance under the condition $\beta a < 1$. As the complete form of the dispersion relation is rather involved, it is justified to seek a satisfactory approximant in the relevant wavenumber range. We have shown that the square-root approximation is suitable to play this role. We also believe that this approximation is of advantage in certain cases of physical interpretation of phenomena, where it leads to greater transparency.

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